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Self-avoiding walks and trails on the 3.12^2 lattice

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Abstract

We find the generating function of self-avoiding walks (SAWs) and trails on a semi-regular lattice called the 3.12^2 lattice in terms of the generating functions of simple graphs, such as SAWs, self-avoiding polygons and tadpole graphs on the hexagonal lattice. Since the growth constant for these graphs is known on the hexagonal lattice we can find the growth constant for both walks and trails on the 3.12^2 lattice. A result of Watson (1970 *J. Phys. C: Solid State Phys.* **3** L28–30) then allows us to find the generating function and growth constant of neighbour-avoiding walks on the covering lattice of the 3.12^2 lattice which is tetra-valent. A mapping into walks on the covering lattice allows us to obtain improved bounds on the growth constant for a range of lattices.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The enumeration of self-avoiding walks and polygons (SAWs and SAPs, respectively) are long-standing combinatorial problems with many connections to statistical mechanics and theoretical chemistry. The vast majority of results for these models on two-dimensional lattices concern SAWs and SAPs embedded in one of the three regular planar lattices, the triangular, the square or the hexagonal lattice.

More recently, attention has turned to other two-dimensional lattices, including semi-regular lattices (also known as Archimedian or homogeneous tilings [9]) quasi-periodic lattices [1] and non-Euclidean hyperbolic lattices [11].

In an earlier paper Jensen and Guttmann [9] studied the number of SAPs on the semi-regular 3.12^2 lattice (see figure 1). The 3.12^2 lattice can be constructed from the hexagonal lattice by replacing each vertex with a triangle as shown in figure 1.

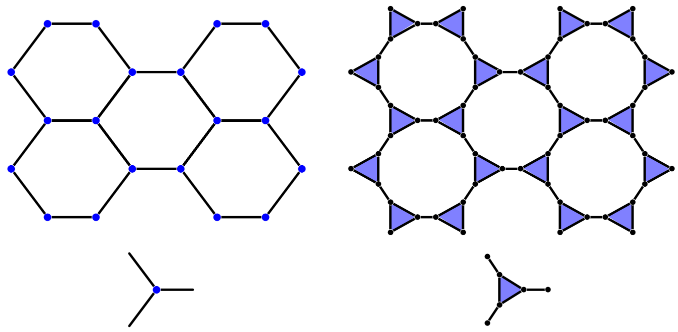

Figure 1. The 3.12^2 lattice (right) can be constructed from the hexagonal lattice (left) by replacing each vertex with a triangle as shown.

While the number of SAWs on this lattice was also discussed in [9] there is an error in their result due to the effects of the endpoints of the walks—this was noted by Alm and Parviainen [3]. In this paper we correct this error and correctly compute the number of SAWs on the 3.12^2 lattice in terms of the number of SAWs (and related graphs) on the hexagonal lattice. Similar reasoning then allows us to extend this result to the number of self-avoiding trails (SATs).

In section 2 we obtain the generating functions for SAWs and SATs on the 3.12^2 lattice. This also gives the growth constant for SAWs on this lattice. In section 3 we give a general bijection between SAWs on a given lattice and SAWs with no nearest-neighbour contacts on the *covering lattice*. From this bijection we find the growth constant of neighbour-avoiding SAW on this new lattice which is tetra-valent.

2. Mapping between the 3.12^2 lattice and the hexagonal lattice

In this section we calculate the generating functions of SAWs and SATs on the 3.12^2 lattice counted by the number of edges. Let these generating functions be $S_w(z)$ and $S_t(z)$ respectively. The first few terms of these generating functions are given in table 1; they were computed using a recursive backtracking algorithm. Not surprisingly, we have been unable to find closed-form expressions for these generating functions. However, by constructing a mapping between the 3.12^2 lattice and the hexagonal lattice we can express $S_w(z)$ and $S_t(z)$ in terms of the generating functions of graphs (counted by the number of edges) embedded in the hexagonal lattice. In particular, we require the following generating functions:

- let $W(z)$ be the generating function of self-avoiding walks, without the constant term. That is to say, the first term is $O(z)$;
- let $R(z)$ be the generating function of returns (or SAPs with a single marked vertex);
- let $T(z)$ be the generating function of tadpoles;
- and finally let $\Theta(z)$, $D(z)$ and $E(z)$ be the generating functions of theta graphs, dumbbells and figure-eights respectively. (Note that figure-eights have a vertex of degree 4, and so are not embeddable in the 3.12^2 or hexagonal lattices.)

Pictures of these graphs are given in figure 2, and the first few coefficients of the corresponding generating functions are given in table 2.

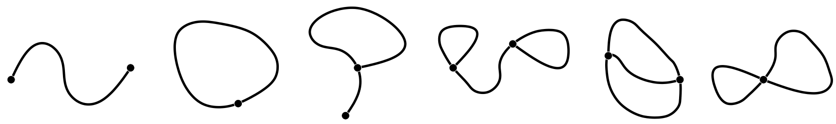

Figure 2. (From left to right) Walks, returns, tadpoles, dumbbells, theta graphs and figure-eights.

Table 1. The first few terms of the generating functions for SAWs and SATs on the 3.12^2 lattice. The data were generated using a backtracking algorithm.

n	SAW	Trails
0	1	1
1	3	3
2	6	6
3	10	12
4	18	22
5	32	40
6	56	72
7	100	128
8	176	224
9	312	400
10	552	704
11	976	1 248
12	1 724	2 208
13	3 018	3 900
14	5 240	6 870
15	9 078	12 002
16	15 780	20 860
17	27 502	36 232
18	47 952	63 086
19	83 602	110 002
20	145 700	191 808
21	253 666	334 388
22	440 696	582 590
23	763 624	1013 674
24	1321 176	1760 024
25	2286 260	3049 440
26	3959 928	5278 204

The main result of the paper is the following theorem:

Theorem 1. *The generating function of self-avoiding walks on the 3.12^2 lattice is given by*

$$S_w(z) = e + \frac{e^2}{3v}W(zv) + \frac{m}{3v}R(zv) + \frac{2de}{3v^2}T(zv) + \frac{2d^2}{3v^3}D(zv) + \frac{d^2}{v^3}\Theta(zv)$$

where $v = z + z^2$, $e = 1 + 2z + 2z^2$, $m = 1 + 2z$ and $d = 2z$.

Similarly, the generating function of self-avoiding trails on the 3.12^2 lattice is given by

$$S_t(z) = e + \frac{e^2}{3v}W(zv) + \frac{m}{3v}R(zv) + \frac{2de}{3v^2}T(zv) + \frac{2d^2}{3v^3}D(zv) + \frac{d^2}{v^3}\Theta(zv)$$

where $v = z + z^2$, $e = 1 + 2z + 2z^2 + 2z^3$, $m = 1 + 4z + 7z^2 + 6z^3$ and $d = 2z + 6z^2$.

Incidentally, our enumeration to order z^{35} is fortunately consistent with the theorem.

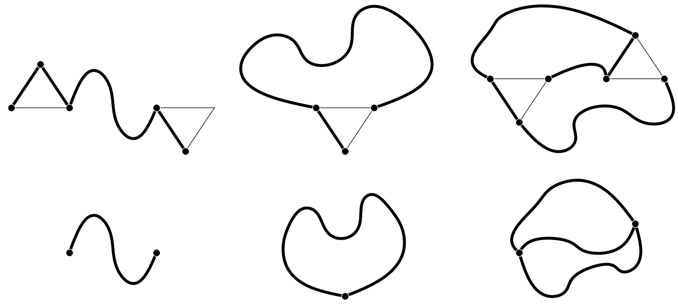

Figure 3. Examples of mapping walks from the 3.12^2 lattice (top) to graphs on the hexagonal lattice (bottom). Depending on the positions of the endpoints of the original SAW it may map to (among other possibilities) a walk, a return or a theta graph.

Table 2. The first few terms of the generating functions of SAWs, returns, tadpoles, dumbbells and theta graphs on the hexagonal lattice. The data were generated by a backtracking algorithm and from data supplied by M F Sykes.

n	SAW	Returns	Tadpoles	Dumbbells	Theta graphs
1	3	0	0	0	0
2	6	0	0	0	0
3	12	0	0	0	0
4	24	0	0	0	0
5	48	0	0	0	0
6	90	6	0	0	0
7	174	0	3	0	0
8	336	0	6	0	0
9	648	0	12	0	0
10	1 218	30	24	0	0
11	2 328	0	54	0	$1\frac{1}{2}$
12	4 416	24	108	0	0
13	8 388	0	222	$1\frac{1}{2}$	0
14	15 780	168	414	3	3
15	29 892	0	834	6	9
16	56 268	288	1 614	9	3
17	106 200	0	3 168	33	12
18	199 350	1 170	5 940	63	24
19	375 504	0	11 598	120	$64\frac{1}{2}$
20	704 304	2 760	21 972	225	54
21	1 323 996	0	42 306	534	147
22	2 479 692	9 504	79 398	975	219
23	4 654 464	0	152 460	1 953	546
24	8 710 212	25 776	286 470	3 672	627
25	16 328 220	0	546 102	$7\,627\frac{1}{2}$	1536
26	30 526 374	84 006	1023 030	14 103	2127

Proof. Let φ be a self-avoiding walk on the 3.12^2 lattice. We map φ to an object ψ on the hexagonal lattice by contracting each triangular face to a vertex while keeping track of the location of the endpoints. Depending on the locations of the endpoints of φ, ψ can be a walk, a return, a tadpole, a dumbbell or a theta graph (see, e.g., figure 3). In order to determine which family of graphs ψ belongs to, we need to consider the possible bond configurations of the

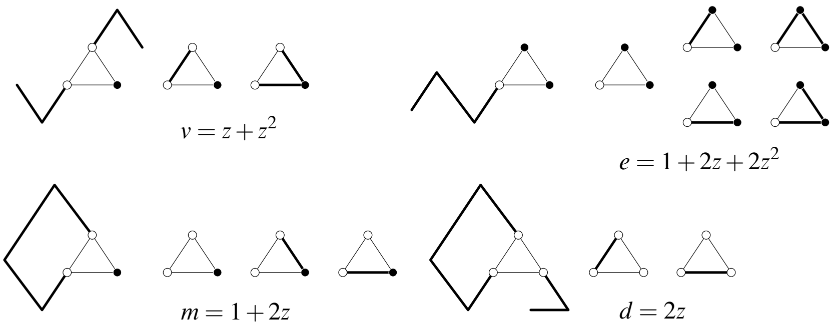

Figure 4. The different types of triangles for the SAW mapping and the possible bond configurations within them.

triangular faces incident upon the original walk φ (see figure 4). In particular if a triangular face is connected to

- one edge of φ, then it is an e-triangle;
- two edges of φ, (which connect outside the triangle) then it is an m-triangle;
- two edges of φ, (that connect via the triangle) then it a v-triangle;
- three edges of φ, then it is a d-triangle.

Since φ contains only two endpoints, the graph ψ must belong to one of only six sets; if φ contains:

- a single e triangle, then ψ is a single vertex;
- two e triangles, then ψ is a walk;
- a single m triangle, then ψ is a return;
- one e and one d triangle, then ψ is a tadpole;
- two d triangles, then ψ is either a dumbbell or a theta graph.

We define a reverse mapping $\psi \mapsto \varphi$ by expanding each vertex on the hexagonal lattice to form a triangle. This is a one-to-many mapping.

- A single vertex becomes an e-triangle.
- A walk of n edges contains 2 vertices that map to e-triangles, and $n-1$ vertices that map to v-triangles.
- A return of n edges contains 1 marked vertex that becomes an m-triangle and $n-1$ vertices that become v-triangles.
- A tadpole of n-edges contains a single degree 1 vertex which becomes an e-triangle and a single degree 3 vertex that becomes a d-triangle. The remaining $n-2$ vertices become v-triangles.
- A theta graph or dumbbell contains two vertices of degree 3 which become d-triangles and the remaining $n-3$ vertices become v-triangles.

These possibilities give rise to each of the terms in the equations in theorem 1. The contributions of the different triangles are given in figure 4—for example the contribution from the case when ψ is a walk gives rise to $\frac{e^2}{3v}W(zv) = \frac{(1+2z+2z^2)^2}{3(z+z^2)}W(z^2+z^3)$. The prefactors arise from the number of orientations and symmetries of the graphs.

The proof for trails is very similar except that the endpoint considerations are a little more complicated. Again a trail φ maps to a graph ψ which is either a single vertex, a walk, a

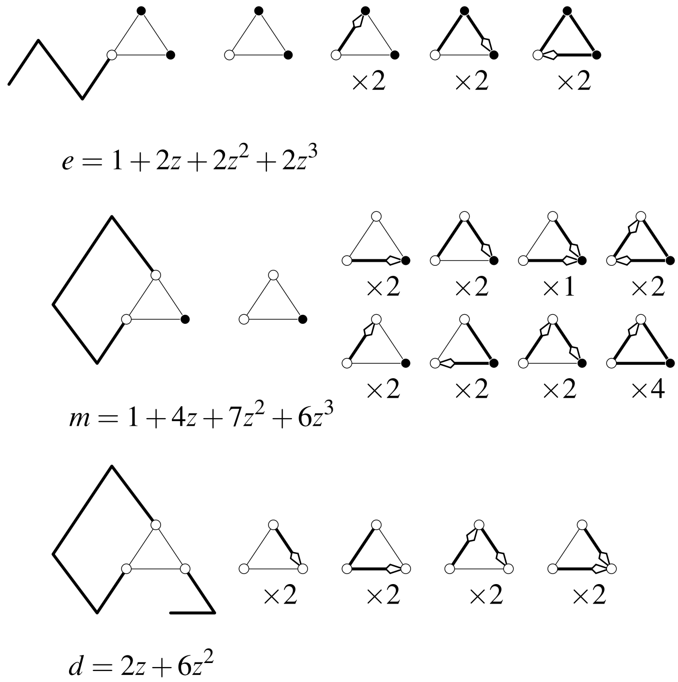

Figure 5. The different types of triangles for the SAT mapping and the possible bond configurations within them.

return, a tadpole, a dumbbell or a theta graph. The contributions of e-, m- and d-triangles are slightly different. See figure 5. □

We can improve these results by using Sykes's counting theorem [12], which relates the SAW generating function to the generating functions of the other families of graphs listed above. This allows us to eliminate some generating functions from the above results. In particular we are able to eliminate either the contribution of tadpoles, or returns, or both theta graphs and dumbbells. We elect to remove the last two.

Theorem 2 (Sykes counting theorem [12])**.** *The generating function of tadpoles on a lattice of coordination number q is given by*

$$T(z) = \frac{z^2(q-1)^2 - z(q-1)}{2} W(z) + \frac{z^2 q(q-1)}{2} - \frac{z}{2} R(z) - 4E(z) - 6\Theta(z) - 4D(z).$$

For the 3.12^2 *lattice,* $q = 3$ *and the result becomes*

$$T(z) = W(z)(2z^2 - z) + 3z^2 - \frac{z}{2} R(z) - 6\Theta(z) - 4D(z).$$

Eliminating the term involving the dumbbell and theta-graph generating functions from theorem 1 gives

Theorem 3. *The generating function of self-avoiding walks on the 3.12*^{2} *lattice is given by*

$$S_w(z) = P_0(z) + P_1(z)W(zv) + P_2(z)R(zv) + P_3(z)T(zv)$$

where $v = z + z^2$ and the $P_i(z)$ are given by

$$P_0(z) = \frac{1 + 3z + 4z^2 + 4z^3}{1 + z}, \qquad P_1(z) = \frac{1 + 5z + 10z^2 + 16z^3 + 16z^4 + 8z^5}{3z(1 + z)^2},$$

$$P_2(z) = \frac{1 + 3z + z^2}{3z(1 + z)^2} \qquad \textit{and} \qquad P_3(z) = \frac{2(1 + 6z + 8z^2 + 4z^3)}{3z(1 + z)^3}.$$

Similarly the generating function of self-avoiding trails on the 3.12^2 lattice is given by

$$S_t(z) = Q_0(z) + Q_1(z)W(zv) + Q_2(z)R(zv) + Q_3(z)T(zv)$$

where $v = z + z^2$ and the $Q_i(z)$ are given by

$$Q_0(z) = \frac{1 + 3z + 4z^2 + 6z^3 + 14z^4 + 18z^5}{1 + z},$$

$$Q_1(z) = \frac{1 + 5z + 10z^2 + 8z^3 + 10z^4 + 48z^5 + 72z^6 + 40z^7}{3z(1 + z)^2},$$

$$Q_2(z) = \frac{1 + 5z + 10z^2 + 7z^3 - 3z^4}{3z(1 + z)^2}$$

and

$$Q_3(z) = \frac{2(1 + 3z)(1 + 3z + 8z^2 + 8z^3 + 4z^4)}{3z(1 + z)^3}.$$

Proof. This follows by eliminating the contributions of the theta graph and dumbbell generating functions between theorems 1 and 2. □

These generating functions then allow us to obtain a number of results on the growth constants for various models on various lattices. The growth constant of self-avoiding walks is defined as the limit

$$\lim_{n\to\infty} c_n^{1/n} = \mu, \tag{1}$$

where c_n is the number of SAWs of length n. Hammersley [6] first proved that this limit exists and is finite. The value of μ is not known *rigorously* on any two-dimensional lattice; however, an argument due to Nienhuis [10] implies that on the hexagonal lattice

$$\lim_{n\to\infty} c_n^{1/n} = \mu = \sqrt{2 + \sqrt{2}}. \tag{2}$$

While this argument is *not entirely rigorous*, the result is supported by strong theoretical and numerical evidence [10, 4] and is widely accepted. More recently Jensen [8] and Alm and Parviainen [3] gave rigorous bounds on the value of μ:

$$1.841\,925 < \mu \approx 1.847\,7590\ldots < 1.868\,832. \tag{3}$$

Using the believed exact value of μ and theorem 1 or 3, we are able to find the growth constant of walks and trails on the 3.12^2 lattice.

Corollary 4. *The growth constant for self-avoiding walks and self-avoiding trails on the 3.12^2 lattice is the largest positive real solution of*

$$\lambda^{12} - 4\lambda^8 - 8\lambda^7 - 4\lambda^6 + 2\lambda^4 + 8\lambda^3 + 12\lambda^2 + 8\lambda + 2 = 0$$

which is

$$\lambda = 1.711\,041\,297\ldots.$$

Note that this relies on Nienhuis' result that $\mu = \sqrt{2 + \sqrt{2}}$ on the hexagonal lattice.

We also note that this agrees with the rigorous bounds given in [8] and [3] of

$$1.708\,553 < \lambda < 1.719\,254$$

Proof. We find the growth constant of SAWs and SATs on the 3.12^2 lattice by examining the radius of convergence of their generating functions. In particular if the radius of convergence is ρ, then the growth constant is $1/\rho$.

Theorem 1 implies that the singularities of $S_w(z)$ and $S_t(z)$ arise from the singularities of $W(z^2+z^3)$, $R(z^2+z^3)$, $T(z^2+z^3)$, $D(z^2+z^3)$ and $\Theta(z^2+z^3)$ and from the factors of $\frac{1}{z(1+z)}$. There is no singularity at $z=0$, since the factors are cancelled by factors of z in the hexagonal lattice series, and the factors of $1/(1+z)$ give singularities at $z=-1$.

It was shown by Guttmann and Whittington [5] that returns, tadpoles, dumbbells and theta graphs all have the same growth constant as self-avoiding walks—their results are for the square lattice, but apply, *mutatis mutandis*, to the hexagonal lattice.

Hence the generating functions $W(x)$, $R(x)$, $T(x)$, $D(x)$ and $\Theta(x)$ all have a dominant singularity at $x=(2+\sqrt{2})^{-1/2}$ and so the generating functions $S_w(z)$ and $S_t(z)$ have dominant singularities at $z=z_c$ satisfying $z_c^2+z_c^3=(2+\sqrt{2})^{-1/2}$. Some manipulation of this equation implies that the growth constant must satisfy the equation given in the statement of the corollary. □

We note that this is the same as the growth constant for self-avoiding polygons on the 3.12^2 lattice given in [9], as might be expected. In fact, Hammersley [7] proved that the growth constants for SAWs and SAPs are equal on the d-dimensional hypercubic lattice, for any given lattice dimensionality d. The proof of this result depends on a simple geometric unfolding argument that becomes extremely messy on the 3.12^2 lattice. As a result, it is unlikely that one would wish to prove equality in that way. The explicit mapping we have given is therefore a more appropriate route to the proof.

3. Covering lattice results

A result of Watson [13] gives a bijection between SAWs on a given lattice $\mathcal{L}$ and *neighbour-avoiding walks* (NAW) on a related lattice called the *covering lattice* $\mathcal{L}^c$. This bijection means that we can also determine the number of NAWs on the covering lattice of the 3.12^2 lattice.

First let us define the above terms. A neighbour-avoiding walk is a SAW with the additional restriction that adjacent occupied vertices must be connected by a single edge of the walk. Given a graph $\mathcal{L}$ its covering graph $\mathcal{L}^c$ is defined as follows:

- For each edge e_i in the edge set of $\mathcal{L}$ create a vertex in $\mathcal{L}^c$.
- If two edges e_i, e_j in the edge set of $\mathcal{L}$ meet at the same vertex then create an edge in $\mathcal{L}^c$ connecting the vertices corresponding to e_i and e_j in $\mathcal{L}^c$.

In figure 6 we show the covering lattice of the square grid, in figure 7 the covering lattice of the 3.12^2 lattice and in figure 8 the covering lattice of the hexagonal lattice. Note that the covering lattices of the 3.12^2 and hexagonal lattices are tetra-valent.

Theorem 5 ([13])**.** *There is a bijection between n-edge self-avoiding walks on the lattice $\mathcal{L}$ and $(n-1)$-edge neighbour-avoiding walks on the covering lattice $\mathcal{L}^c$.*

Proof. A self-avoiding walk of n-edges may be encoded as a list of edges $(e_1, e_2, \ldots, e_n)$, such that $e_i \neq e_j$ (for $i \neq j$) and e_i and e_j meet at a vertex if and only if $i = j \pm 1$. Mapping these edges to the corresponding vertices on the covering lattice we obtain a list of vertices

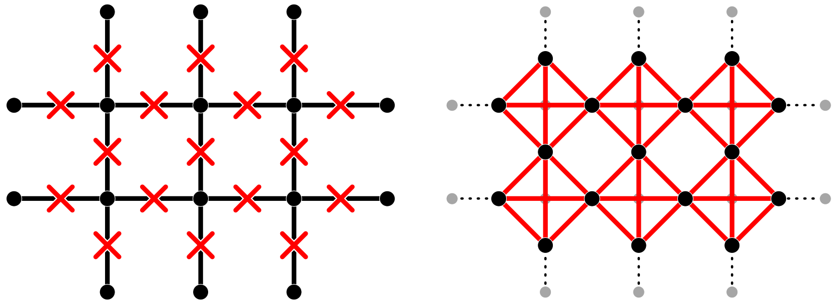

Figure 6. (left) A portion of the square lattice with a vertex placed at each edge (marked with a cross) and (right) the corresponding portion of the covering lattice of the square lattice—each vertex of the covering lattice corresponds to an edge in the square lattice, and two vertices are connected by an edge if the corresponding edges in the square lattice meet at a vertex.

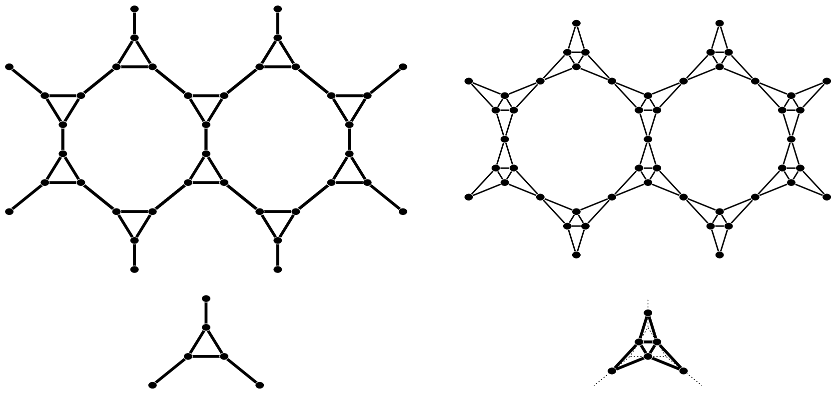

Figure 7. A portion of the 3.12^2 lattice and the corresponding portion of its covering lattice. Below each lattice we show a unit cell of the 3.12^2 lattice and the corresponding cell in the covering lattice.

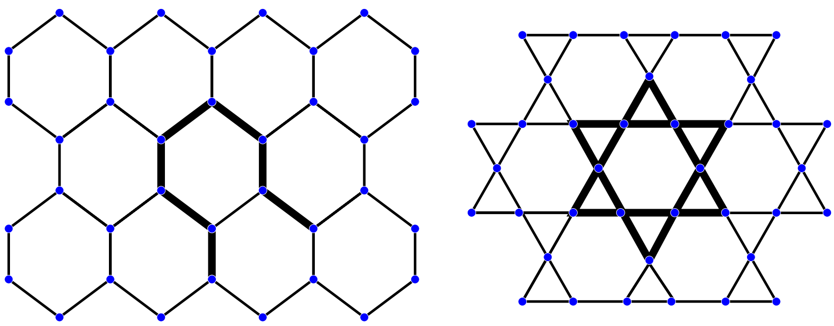

Figure 8. (left) A self-avoiding walk on the hexagonal lattice is shown with bold lines. (right) The image set of this walk on the covering 3.6.3.6 lattice is shown with bold lines.

$(v_1, v_2, \ldots, v_n)$ such that $v_i \neq v_j$ and such that v_i and v_j are joined by an edge if an only if $i = j \pm 1$. This is just an $(n-1)$-edge neighbour-avoiding walk on the covering lattice. $\square$

Corollary 6. *The growth constant for neighbour-avoiding walks on the covering lattice of the 3.12^2 lattice is equal to that of SATs and SAWs on the 3.12^2 lattice and is*

$$\mu = 1.711\,041\,297\ldots$$

3.1. Mapping onto SAWs and into SATs

For any tri-valent lattice $\mathcal{L}$, we can define a mapping from the set of SAWs, to a set V of walks on the covering lattice $\mathcal{L}^c$.

For a given self-avoiding walk on $\mathcal{L}$, with edge-sequence $\{e_1, e_2, \ldots, e_n\}$, map each edge e_i, except the last, to two walks on the covering lattice $\mathcal{L}^c$, one being a single edge of a triangle, the second being the other two edges. Thus one is of length 1: the edge in $\mathcal{L}^c$ that connects e_i and e_{i+1}, and the other is of length 2: the two-stepped self-avoiding walk in $\mathcal{L}^c$ that connects e_i and e_{i+1}. The image set $V = V(\mathcal{L}^c)$ of the set of self-avoiding walks on $\mathcal{L}$ under this mapping is a set of walks on $\mathcal{L}^c$. An example of the image set of a self-avoiding walk under this mapping is shown in figure 8.

Let W and T denote the set of, respectively, SAWs and SATs on $\mathcal{L}^c$. To make all walks in the image set start at the same vertex, corresponding to the edge e in $\mathcal{L}$, assume that all self-avoiding walks on $\mathcal{L}$ start with the edge e. In particular, all have at least one edge.

Lemma 7. *It follows that $W \subseteq V \subseteq T$.*

Proof. We first show that the mapping is into SAT. Each subwalk e_i, v_i, e_{i+1} of a self-avoiding walk w on $\mathcal{L}$ uniquely determines a triangle, corresponding to the vertex v_i, and one or two edges of that triangle that appear in the image walk. As all vertices in w are distinct, no edge in $\mathcal{L}^c$ can appear twice in the image walk. For the other inclusion, consider a self-avoiding walk $\tilde{w} = \{e_1, e_2, \ldots, e_n\}$ on $\mathcal{L}^c$. Decompose $\tilde{w}$ into subwalks by the following rule. Set $i = 1$ and $k = 1$.

Let $W_k = \{e_i, e_{i+1}\}$ if e_i and e_{i+1} are parts of different triangles, and let $i = i + 2$ and $k = k + 1$. Otherwise let $W_k = \{e_i\}$, and let $i = i + 1$ and $k = k + 1$. Repeat.

Each subwalk W_k corresponds to a unique triangle in $\mathcal{L}^c$, and so to a unique vertex v_k in $\mathcal{L}$. It follows from the construction that v_k is adjacent to v_{k-1} and v_{k+1} (if applicable). Let w be the walk with vertex sequence $\{v_1, v_2, \ldots, v_K\}$, where K is the index of the last subwalk. Since at most two edges (which must then be adjacent in the walk) of a SAW $\tilde{w}$ can be part of any given triangle, the walk w is self-avoiding. □

Letting w_n denote the number of different walks of length n in V, we may thus define a growth constant $\omega = \lim_{n\to\infty} w_n^{1/n}$, and a generating function $V(z) = \sum_n w_n z^n$, with radius of convergence $z_c = 1/\omega$. By the lemma, it follows immediately that

$$\mu_W(\mathcal{L}^c) \leqslant \omega(\mathcal{L}) \leqslant \mu_T(\mathcal{L}^c),$$

where μ_W and μ_T are the growth constants for SAWs and SATs, respectively. Let $L(z)$ be the generating function for SAWs on $\mathcal{L}$ (with the above restriction on the first step). Since $(z + z^2)V(z) = L(z + z^2)$, we have $\mu^{-1} = \omega_W(\mathcal{L})^{-1} + \omega_W(\mathcal{L})^{-2}$. An immediate consequence of this equation is the following theorem:

Theorem 8. *Let $\mu = \mu_W(\mathcal{L})$.*

$$\omega = \tfrac{1}{2}(\mu + \sqrt{4\mu + \mu^2}).$$

3.1.1. The hexagonal lattice. It is well known that the covering lattice of the hexagonal lattice is the Kagomé, or 3.6.3.6, lattice. Using Nienhuis' value $\mu_W(6^3) = \mu = \sqrt{2+\sqrt{2}}$ we find

$$\mu_W(3.6.3.6) \leqslant \omega(3.6.3.6) = 2.567\,4465\ldots \leqslant \mu_{SAT}(3.6.3.6).$$

Using the rigorous bounds $1.841\,925 < \mu_W(6^3) < 1.868\,832$, due to [8] and [3] respectively, we get

$$\mu_W(3.6.3.6) \leqslant \omega(3.6.3.6) < 2.590\,3041,$$

and

$$2.561\,114 < \omega(3.6.3.6) \leqslant \mu_T(3.6.3.6).$$

These values should be compared with the estimate $\mu_W(3.6.3.6) \approx 2.560\,576\,765$, from [8], and the previous best upper bound $\mu_W(3.6.3.6) \leqslant 2.604\,93$, from [2].

3.1.2. The 3.12^2 lattice. Using the Nienhuis value, we get bounds for the covering lattice $(3.12^2)^c$ as follows:

$$\mu_W((3.12^2)^c) \leqslant 2.418\,5167\ldots \leqslant \mu_T((3.12^2)^c),$$

and using the rigorous bounds given in [8] and [3],

$$\mu_W((3.12^2)^c) < 2.427\,4958 \qquad \text{and} \qquad 2.415\,7954 < \mu_T((3.12^2)^c).$$

Remark 9. An analogous mapping may be defined for lattices with higher coordination number. However, as the coordination number increases, the difference between the SAW and SAT growth constants increases.

Remark 10. In a similar way, an improved upper bound for the dual $D(3.12^2)$ of the 3.12^2 lattice may be derived. Note that the triangular lattice 3^6 is a subgraph of $D(3.12^2)$. Map each edge e in a SAW on 3^6 to three walks on $D(3.12^2)$: the edge e itself, and the two walks of length 2 that connect the endpoints of the edge e. The image set I of the mapping contains all self-avoiding walks on $D(3.12^2)$. The growth constants for these two sets are related by $\mu_W(3^6)^{-1} = \mu_I(D(3.12^2))^{-1} + 2\mu_I(D(3.12^2))^{-2}$. Using the upper bound $\mu_W(3^6) < 4.25142$, [2], we get $\mu_W(D(3.12^2)) < 5.734\,24$, a slight improvement of the previous upper bound, 5.796 21, [2].

4. Conclusions

We have found an expression for the number of self-avoiding walks and self-avoiding trails on the 3.12^2 lattice in terms of the generating functions of SAWs, returns and tadpoles on the hexagonal lattice.

A bijection allows us to then express number of neighbour-avoiding walks on the covering lattice of the 3.12^2 lattice in terms of these same objects on the hexagonal lattice. Since the growth constant of walks is known (non-rigorously) on the hexagonal lattice, our expressions also give the growth constant for SAWs and SATs on the 3.12^2 lattice and NAWs on the covering lattice. This last lattice is tetra-valent.

A similar mapping, which is *into* the set of SATs and *onto* the set of SAWs on the covering lattice, gives bounds for the growth constants on the 3.6.3.6 lattice and the covering lattice of the 3.12^2 lattice.

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